Understanding contrasts

Dr. Martyn McFarquhar

Division of Neuroscience & Experimental Psychology
The University of Manchester

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What are contrasts?

Informally, a contrast is a method of asking a question about your model in mathematical terms.

The GLM partitions the data into model and error.

\[ Y = \underbrace{X\beta}_{\text{model}} + \underbrace{\epsilon}_{\text{error}} \]

The model is the typical value of the data for given values of the predictor variables.

This is formed via a linear combination of known predictor variables and unknown parameters.

As the parameters represent the relationship between the predictors and the outcome, their estimated values are the part of the model that we are most interested in.
Formal definition

General Linear Hypothesis

All hypothesis testing in the GLM is based on the following

\[ H_0 : L\beta = m \]

\( L \) is the \( m \times k \) matrix of weights
\( \beta \) is the \( k \times 1 \) vector of model parameters
\( m \) is the \( m \times 1 \) vector of proposed values

The hypothesis is that some linear combination(s) of the parameters equals some proposed value(s)

In SPM the matrix \( m \) is fixed to contain all zeros — each linear combination of the parameters in the rows of \( L \) are equal to 0
Formal definition

General Linear Hypothesis

All hypothesis testing in the GLM is based on the following

$$\mathcal{H}_0 : L\beta = m$$

This is a very flexible system

$$\mathcal{H}_0 : \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = 0$$  \(\beta_1\) is equal to 0

$$\mathcal{H}_0 : \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$  \(\beta_1\) or \(\beta_2\) is equal to 0

$$\mathcal{H}_0 : \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = 0$$  the difference between \(\beta_1\) and \(\beta_2\) is 0

$$\mathcal{H}_0 : \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = 0$$  the average of \(\beta_1\) and \(\beta_2\) is 0
Parameters and interpretation

As **contrasts** are **linear combinations** of parameters, their **interpretation** depends on understanding the parameters.

1st-level

The parameters **scale** the **predicted shape of the response** to **best fit the data**.

Their values tell us about the **magnitude** and **direction** of the **average change from baseline** in each experimental condition.
Parameters and interpretation

As contrasts are linear combinations of parameters, their interpretation depends on understanding the parameters.

2nd-level

Indicator variables
- Contain only 1 or 0 to model a factor
- Parameters are cell means

Continuous covariates
- Contain any value within a range
- Parameters are regression slopes

Comparing cell means is sensible, but comparing regression slopes only makes sense if the predictors are scaled identically.
Parameters and interpretation

For multiple predictor variables, tests on each parameter is interpreted as testing the unique variability explained after adjusting for all other variables.

Effect of predictor 1
Parameters and interpretation

For multiple predictor variables, tests on each parameter is interpreted as testing the unique variability explained after adjusting for all other variables.

Effect of predictor 2
Parameters and interpretation

For **multiple** predictor variables, tests on each parameter is interpreted as testing the **unique variability** explained **after adjusting for all other variables**

For **contrasts**, this means that even if a parameter is given a **weight of 0**, the other parameters are **still adjusted** for its presence

\[ L = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \]

**ANCOVA**

The individual **cell means** are still **adjusted** for the **covariate**

The value of the **parameter estimates** depends on **other variables in the model**

**Contrasts** ask questions about the model — they **do not alter the model**
Parameters and interpretation

Parameters are estimated using

$$\hat{\beta} = (X'X)^{-1}X'Y$$

Notice that this depends on the design matrix

Important consequences

1. The estimates depend on the scaling of the columns of $X$
2. If $X$ is rank deficient (contains redundant columns) then there are an infinite number of solutions

If $X$ is rank deficient then $(X'X)^{-1}$ does not exist

We can still find a solution using a pseudo-inverse — creates complications in specifying and interpreting contrasts
Estimable functions

An \( L \) matrix defines an **estimable function** of the model parameters if it can be expressed as

\[
L = TX
\]

Our contrast is only **meaningful** if it is formed from a **linear combination** of the **rows** of the **design matrix**

An **estimable contrast** is also one that can be expressed as a **linear combination** of the **model estimates**

\[
\hat{c} = L\hat{\beta} = TX\hat{\beta} = T\hat{Y}
\]

The **model estimates** dictate how **meaningful** values are formed from combining the **predictors** and the **parameters**

A question is only meaningful if it **respects this combination**
Estimable functions

Example - overparameterised 2x2 ANOVA

\[ y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk} \]

Here the ANOVA effects are explicit in the model.

- Constant
- Main effect of A
- Main effect of B
- A x B interaction
Estimable functions

Example - overparameterised 2x2 ANOVA

The **design matrix** is **rank-deficient** so estimation requires use of a **pseudo-inverse**

\[ \hat{\beta} = (X'X)^+X'Y \]

The parameters have a **degree of arbitrariness** to their values — certain combinations are **still meaningful**

**Find 2 number that sum to 5**

\[ a + b = 5 \]

- \( a = 2, \ b = 3 \)
- \( a = -6.37, \ b = 11.37 \)
- \( a = 0.4, \ b = 4.6 \)
- \( a = 4322, \ b = -4317 \)

There are **infinite choices** — the individual values are **arbitrary** but the sum **is not**
Estimable functions

Example - overparameterised 2x2 ANOVA

Contrasts must respect the meaningful combinations of parameters given by the rows of $X$

Main effect of $A$?

$L = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times \times$

$L_{A1B1} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$
$L_{A1B2} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$
$L_{A2B1} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$
$L_{A2B2} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

$L_{A1} = \begin{bmatrix} 1 & 1 & 0 & 0.5 & 0.5 & 0.5 & 0.5 & 0 \end{bmatrix}$
$L_{A2} = \begin{bmatrix} 1 & 0 & 1 & 0.5 & 0.5 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}$

Correct main effect of $A$

$L_{A1-A2} = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0.5 & 0.5 & -0.5 & -0.5 \end{bmatrix}$
Estimable functions

Estimable functions in SPM

Non-estimable parameters are indicated by a grey box below the associated column.
Estimable functions

Estimable functions in SPM

To make sure that the contrast we are testing is **estimable**, SPM will perform an **estimability test**.

This ensures that we are testing combinations that **do not depend** on the solution for the parameters (see McFarquhar, 2016).
Estimable functions

Estimable functions in SPM

\[ T = LX^+ \]

\[ X = \begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \]

If \( L \) is an \textbf{estimable function} \[ L = TX \]

\[ L = \begin{bmatrix}
0 & 1 & -1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}
\end{bmatrix} \]

\[ T = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}
\end{bmatrix} \]

\[ TX = \begin{bmatrix}
0 & 1 & -1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}
\end{bmatrix} \]

If \( L \) is not an \textbf{estimable function} \[ L \neq TX \]

\[ L = \begin{bmatrix}
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \]

\[ T = \begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3}
\end{bmatrix} \]

\[ TX = \begin{bmatrix}
0 & 2 & -2 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3}
\end{bmatrix} \]
Contrast interpretation

Most of the time we work with well parameterised models and so do not have to worry about estimability — check the grey boxes!

Despite this, a contrast can be estimable but misinterpreted

This largely comes down to
1. Understanding what the parameters mean
2. Making sure that the contrast is testing what you think it is

You should always ensure that you are clear how to interpret the parameters and contrasts from your model

How can you know what your results mean of you don’t understand what the model is telling you, or what your questions are?
Contrast interpretation

1st-level example — baselines

**Model 1**
(A, B, rest, const.)

- $\mathbf{L}_{A-B} = [1 \ -1 \ 0 \ 0]$
- $\mathbf{L}_{A-R} = [1 \ 0 \ -1 \ 0]$
- $\mathbf{L}_{B-R} = [0 \ 1 \ -1 \ 0]$
- $\mathbf{L}_A = [1 \ 0 \ 0 \ 1]$
- $\mathbf{L}_B = [0 \ 1 \ 0 \ 1]$

**Model 2**
(A, B, const.)

- $\mathbf{L}_{A-B} = [1 \ -1 \ 0]$
- $\mathbf{L}_{A-R} = [1 \ 0 \ 0]$
- $\mathbf{L}_{B-R} = [0 \ 1 \ 0]$
- $\mathbf{L}_A = [1 \ 0 \ 1]$
- $\mathbf{L}_B = [0 \ 1 \ 1]$

An *explicit* baseline over an *implicit* baseline alters the *interpretation* of the parameters and the contrasts.
Contrast interpretation

2nd-level example — \( t \)-test

**Model 1**  
(A, B, const.)

\[
\begin{align*}
L_{A-B} &= [1, -1, 0] \\
L_A &= [1, 0, 1] \\
L_B &= [0, 1, 1] \\
L_{\text{mean}} &= [0.5, 0.5, 1]
\end{align*}
\]

**Model 2**  
(A, const.)

\[
\begin{align*}
L_{A-B} &= [1, 0] \\
L_A &= [1, 1] \\
L_B &= [0, 1] \\
L_{\text{mean}} &= [0.5, 1]
\end{align*}
\]

**Model 3**  
(A, B)

\[
\begin{align*}
L_{A-B} &= [1, -1] \\
L_A &= [1, 0] \\
L_B &= [0, 1] \\
L_{\text{mean}} &= [0.5, 0.5]
\end{align*}
\]

Different parameterisations lead to the same predicted values, but change the meaning of the parameters.

Need to be clear on what the parameters mean in order to correctly interpret a contrast.
Test statistics

Assuming our contrast is estimable and we are clear on what it means we can test the estimated value against a proposed population value by forming a test statistic.

In SPM we have two options for how to test our contrast value — as a t-contrast or an F-contrast.

Each type is used in different contexts and it is important to understand their differences, and similarities, to ensure you use the most appropriate method for your questions.
$t$-contrasts

In SPM a $t$-contrast is defined by

- An $L$ matrix with a **single row**
- Hypothesis testing using a $t$-statistic
- **One-tailed** $p$-values

$$t = \frac{L\hat{\beta} - m}{\hat{\sigma}\sqrt{L(X'X)^{-1}L'}}$$

Notice that $L$ appears in the **numerator** and **denominator** — scaling of $L$ does not matter

All lead to the same $t$-statistic:

$$L = \begin{bmatrix} 1 & -1 \end{bmatrix} \quad L = \begin{bmatrix} 2 & -2 \end{bmatrix} \quad L = \begin{bmatrix} 1000 & -1000 \end{bmatrix}$$
In SPM a $t$-contrast is defined by

- An $L$ matrix with a single row
- Hypothesis testing using a $t$-statistic
- One-tailed $p$-values

The $p$-values are upper-tail values — you will only see results for positive $t$-statistics

Upper-tail $p$ for $-1.24$ = 0.920
**t-contrasts**

In **SPM** a $t$-contrast is defined by

- An $L$ matrix with a **single row**
- Hypothesis testing using a $t$-statistic
- **One-tailed** $p$-values

The $p$-values are **upper-tail** values — you will only see results for **positive** $t$-statistics

- **Upper-tail** $p$ for $-1.24$ = 0.920
- **Upper-tail** $p$ for 1.24 = 0.08
$t$-contrasts

In SPM a $t$-contrast is defined by
- An $L$ matrix with a **single row**
- Hypothesis testing using a $t$-statistic
- **One-tailed** $p$-values

The $p$-values are **upper-tail** values — you will only see results for
**positive** $t$-statistics

A **positive** $t$-statistic occurs when the direction of the effect
matches the direction of the contrast

\[
L = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{Only see positive effects}
\]

\[
L = \begin{bmatrix} -1 & 0 \end{bmatrix} \quad \text{Only see negative effects}
\]

**One-tailed** $p$-values only suitable for **strong directional hypotheses**
**F-contrasts**

In **SPM** a *F*-contrast is defined by

- An **L** matrix with **multiple rows**
- Hypothesis testing using an *F*-statistic
- **Two-tailed** *p*-values

\[
F = \frac{(L\hat{\beta} - m)'(L(X'X)^{-1}L')^{-1}(L\hat{\beta} - m)}{r\hat{\sigma}^2}
\]

The **numerator** forms a **sum-of-squares** — divided by *r* to form a **mean square**

**Multiple rows** can be thought of as an **OR** question

An *F*-contrast with a **single row** is the same as *t*² — in SPM this allows for a **two-tailed** alternative to a *t*-contrast
F-contrasts

**Simple example**: 1-way ANOVA with a 3-level factor

The **weights** for an F-contrast testing the **main effect** of the factor are:

\[ L = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \]

**Does this seem odd at all?**

Should it not be?

\[ L = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \]
**F-contrasts**

**Simple example:** 1-way ANOVA with a 3-level factor

Should it not be?

\[ L = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \]

This row is redundant

This last row is the **sum** of the other rows — value from this row is **not independent of the other rows**

\[ \beta = \begin{bmatrix} 10 \\ 15 \\ 12 \end{bmatrix} \]

\[ L\beta = \begin{bmatrix} -5 \\ 3 \\ -2 \end{bmatrix} \]

\[ (-5) + 3 = -2 \]

The last row **provides no more information** — connection with **numerator degrees of freedom**
Do the weights have to sum to zero?

A potential source of confusion relates to whether the weights of the contrast must sum to zero.

Distinction between a linear combination and a contrast:

- A contrast is specifically about comparing parameters.
- A linear combination is more general, e.g., averaging, summing etc.

In SPM the term contrast is used generically for either.

In the statistics literature, a contrast has the additional requirement that the weights sum to zero — not necessary for a linear combination.
Do the weights have to sum to zero?

$$\mathcal{H}_0 : \beta_1 - \beta_2 = 0$$

$$L = \begin{bmatrix} 1 & -1 \end{bmatrix}$$ \quad \sum l_i = 0 \quad \checkmark$$

$$L = \begin{bmatrix} 2 & -2 \end{bmatrix}$$ \quad \sum l_i = 0 \quad \checkmark$$

$$L = \begin{bmatrix} 1 & -2 \end{bmatrix}$$ \quad \sum l_i = -1 \quad \times$$

$$\mathcal{H}_0 : (\beta_1 + \beta_2)/2 - \beta_3 = 0$$

$$L = \begin{bmatrix} 0.5 & 0.5 & -1 \end{bmatrix}$$ \quad \sum l_i = 0 \quad \checkmark$$

$$L = \begin{bmatrix} 1 & 1 & -2 \end{bmatrix}$$ \quad \sum l_i = 0 \quad \checkmark$$

$$L = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$$ \quad \sum l_i = 1 \quad \times$$

In both cases we are looking at differences of parameters — these are true contrasts.
Do the weights have to sum to zero?

The **alternative** is when we look at **individual parameters** or **averages of parameters**

\[ H_0 : \beta_1 = 0 \]

\[ \mathbf{L} = \begin{bmatrix} 1 & 0 \end{bmatrix} \sum l_i = 1 \checkmark \]

\[ \mathcal{H}_0 : (\beta_1 + \beta_2)/2 = 0 \]

\[ \mathbf{L} = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \sum l_i = 1 \checkmark \quad \mathbf{L} = \begin{bmatrix} 1 & 1 \end{bmatrix} \sum l_i = 2 \checkmark \]

If the parameters are **estimable** then these are all valid

**Interpretation** must be done with **care** — depends on how the model is **parameterised** (e.g. **implicit** vs **explicit** baselines)
Basic contrasts in ANOVA models

Often, it is the **group-level** where our **hypotheses** are focussed.

Typically, data collected from **factorial designs** will be analysed using an **ANalysis Of VAriance** model.

For designs with >2 factors, the ANOVA model has a number of effects that we may be interested in — **main effects**, **interactions**, **simple effects**.

Important to understand how these are tested using **contrasts**.

SPM **defaults** to a **cell means** ANOVA rather than an **overparameterised** ANOVA — no need to worry about **estimability**.
Basic contrasts in ANOVA models

2 x 2 ANOVA

Cell means are the means from the intersection of the factors and the marginal means are the means from across a factor.

<table>
<thead>
<tr>
<th>Factor B</th>
<th>Factor A</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\mu_{11}$</td>
</tr>
<tr>
<td>2</td>
<td>$\mu_{12}$</td>
</tr>
<tr>
<td></td>
<td>$\mu_{1.}$</td>
</tr>
</tbody>
</table>

Cell means

Marginal means for Factor A

Marginal means for Factor B

The dot notation indicates a subscript averaged over.
Basic contrasts in ANOVA models

2 x 2 ANOVA

The simplest model is the **cell means** model

\[ y_{ijk} = \mu_{ij} + \epsilon_{ijk} \]

Here the **parameters** are the **cell means** of design

The **ANOVA effects** are then formed using **contrasts of the cell means**

- **Main effect of A**  \[ L = [1 \ 1 \ -1 \ -1] \]
- **Main effect of B**  \[ L = [1 \ -1 \ 1 \ -1] \]
- **A x B interaction**  \[ L = [1 \ -1 \ -1 \ 1] \]
Basic contrasts in ANOVA models

2 x 2 ANOVA

The interaction contrast

We are looking for a difference of two differences

\[ A \times B = (A_{1B1} - A_{2B1}) - (A_{1B2} - A_{2B2}) \]

Effect of A at the first level of B

Effect of A at the second level of B
Basic contrasts in ANOVA models

2 x 2 ANOVA

The interaction contrast

We are looking for a difference of two differences

\[ A \times B = (\mu_{11} - \mu_{21}) - (\mu_{12} - \mu_{22}) \]
Basic contrasts in ANOVA models

2 x 2 ANOVA

The interaction contrast

We are looking for a difference of two differences

\[
A \times B = (\mu_{11} - \mu_{21}) - (\mu_{12} - \mu_{22})
\]

\[
L = \begin{bmatrix}
1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1
\end{bmatrix}
\]

The weights for the interaction come from subtracting the weights for the simple effects
Basic contrasts in ANOVA models

2 x 2 ANOVA

The interaction contrast

We are looking for a difference of two differences

\[ A \times B = (\mu_{11} - \mu_{21}) - (\mu_{12} - \mu_{22}) = \mu_{11} - \mu_{21} - \mu_{12} + \mu_{22} \]

\[ L = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix} \]

Alternatively, we can remove the brackets from the expression for the interaction
Basic contrasts in ANOVA models

2 x 2 ANOVA

The interaction contrast

We are looking for a difference of two differences

\[ A \times B = (\mu_{11} - \mu_{21}) - (\mu_{12} - \mu_{22}) \]

\[ = \begin{bmatrix} \mu_{11} & -\mu_{21} & -\mu_{12} & + \mu_{22} \end{bmatrix} \]

\[ L = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix} \]

Alternatively, we can remove the brackets from the expression for the interaction
Advanced contrasts in ANOVA models

Unbalanced designs

In other statistical packages, we have a choice of sums-of-squares when dealing with unbalanced ANOVA designs with >2 factors — Type I, Type II and Type III.

The equally weighted means contrasts we use in SPM are equivalent to Type III sums-of-squares.

There is debate about how sensible the Type III tests are — some authors suggest we should be using Type II instead (e.g. Venables, 1998; Langsrud, 2003; Fox, 2008; Fox and Weisberg, 2011).
Advanced contrasts in ANOVA models

**Type I**

**Sequential** sums of squares where each effect is tested after adding to the model

\[ y_{ijk} = \mu + \epsilon_{ijk} \]
\[ y_{ijk} = \mu + \alpha_i + \epsilon_{ijk} \]

**Type II**

Respects the principle of marginality where main effects are tested assuming any interaction is 0

\[ y_{ijk} = \mu + \beta_j + \epsilon_{ijk} \]
\[ y_{ijk} = \mu + \alpha_i + \beta_j + \epsilon_{ijk} \]

**Type III**

Ignores the principle of marginality so main effects are tested after correcting for interactions

\[ y_{ijk} = \mu + \beta_j + (\alpha \beta)_{ij} + \epsilon_{ijk} \]
\[ y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha \beta)_{ij} + \epsilon_{ijk} \]
Advanced contrasts in ANOVA models

Implementation using contrasts

Both **Type I** and **Type II** are tricky to calculate **weights** for — no facility in **SPM** to do this for us

\[
L_I = \begin{bmatrix}
0 & 1 & -1 & -0.167 & 0.167 & 0.5 & 0.5 & -0.667 & 0.333
\end{bmatrix}
\]

\[
L_{II} = \begin{bmatrix}
0 & 1 & -1 & 0 & 0 & 0.6 & 0.4 & -0.6 & -0.4
\end{bmatrix}
\]

\[
L_{III} = \begin{bmatrix}
0 & 1 & -1 & 0 & 0 & 0.5 & 0.5 & -0.5 & -0.5
\end{bmatrix}
\]

Not particularly **intuitive** when written down — both **Type I** and **Type II** weights depend on the **number of subjects in each cell**

See McFarquhar (2016) for **MATLAB code** and **more detailed discussion** on these approaches
Other uses of contrasts

Contrasts are **generic methods** of selecting **linear combinations** of the **model parameters**

Contrasts are also used in SPM for

- Selecting parameters to **plot** in **bar charts**
- Partitioning the model to **plot** only **effects of interest**

Understanding how this works is **crucial** to getting the most out of the SPM plotting facilities
Plotting results with contrasts

Rules for making plots using contrasts
1. Each row will be a separate bar in the plot
2. Height of the bar is the combination of parameters in that row

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0.5 & 0.5
\end{bmatrix}
\] Bar 1: value of parameter 1
Bar 2: average of parameters 2 and 3

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] Bar 1: value of parameter 1
Bar 2: value of parameter 2
Bar 3: value of parameter 3

\[
\begin{bmatrix}
1 & -1 & 0
\end{bmatrix}
\] Bar 1: difference between parameters 1 and 2
Plotting results with contrasts

2 x 2 ANOVA

Age (older/younger) x Diagnosis (control/depressed)

To plot the interaction we want to select each cell mean

\[
\text{Plot } A \times B = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

4 bars for the cell means
Plotting results with contrasts

2 x 2 ANOVA

Age (older/younger) x Diagnosis (control/depressed)

To plot the main effects we average over the cells of the other factor to form the marginal means

\[
\text{Plot Age} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}
\]

2 bars each averaged across Diagnosis
Plotting results with contrasts

2 x 2 ANOVA

**Age** (older/younger) x **Diagnosis** (control/depressed)

To plot the **main effects** we average over the cells of the **other factor** to form the **marginal means**

$$\text{Plot Diagnosis} = \begin{bmatrix} 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \end{bmatrix}$$

2 bars each averaged across Age
Partitioning effects with contrasts

When attempting to **visualise** effects in our model, sometimes it can be difficult to see what is going on because the **effect of interest** may be **obscured** by another **effect/confound**.

![Graph showing A1B1, A1B2, A2B1, A2B2 categories with comparison values.]

**Clear main effect of A** — is there a **main effect of B**?

At present, any effect of B is **obscured** by the effect of A.
Partitioning effects with contrasts

We can use the contrast for the effect of B to remove from the data those effects that are not related to the effect of B

\[
L = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}
\]

\[
L_0 = I_4 - L'(L')^+
\]

\[
X_0 = XL_0
\]

\[
R_0 = I_n - X_0 X_0^+
\]

\[
Y_{\text{adj}} = R_0 Y
\]

We use the contrast to partition the model into effect of interest and effects of no interest and use this to remove the effects of no interest from the data.

This is the model fit for the partition of the effects of interest + error.
Partitioning effects with contrasts

We can use the contrast for the effect of B to remove from the data those effects that are not related to the effect of B

\[
\begin{align*}
L &= \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \\
L_0 &= I_4 - L'(L')^+ \\
X_0 &= XL_0 \\
R_0 &= I_n - X_0X_0^+ \\
Y_{adj} &= R_0 Y
\end{align*}
\]

We use the contrast to partition the model into effect of interest and effects of no interest and use this to remove the effects of no interest from the data.

This is the model fit for the partition of the effects of interest + error.

Subtracting the residuals gives us the model fit.
Partitioning effects with contrasts

We can use the contrast for the effect of B to remove from the data those effects that are not related to the effect of B.

\[ L = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \]

\[ L_0 = I_4 - L'(L')^+ \]

\[ X_0 = XL_0 \]

\[ R_0 = I_n - X_0X_0^+ \]

\[ Y_{adj} = R_0Y \]

We use the contrast to partition the model into effect of interest and effects of no interest and use this to remove the effects of no interest from the data.

This is what SPM does when you plot fitted responses.
Summary

The use of **contrast weights** allows us to defining **linear combinations** of the parameters estimates.

Typically this is used to define a **hypothesis** about the parameters — important to understand **what** the parameters **mean**.

**Contrasts** must respect the **meaningful** combinations of parameters **dictated** by the rows of $X$ — **only** an issue with **rank-deficient** designs.

**Contrasts** are **flexible** as they can:
- Provide alternative means of testing hypotheses in **unbalanced** designs
- Select parameters for **plotting**
- Partition the design into **effects of interest/no interest**
References

